The Effective Radius Model for Multi-hop Wireless Networks

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Abstract. In this paper, we introduce a novel model, termed as *Effective Radius* (ER), to calculate the expected number of t-hop neighbors in a multi-hop wireless network with a uniform node distribution on the average. This ER model is an analytical tool that recursively computes a t-hop effective radius for $t = 2, 3, \cdots$. The total number of nodes covered by the disk with a t-hop effective radius equals to the expected number of nodes reachable through at most t hops in the original physical topology. We conduct extensive simulation studies to validate our model and the results demonstrate that the ER model is accurate and can be adaptive to different deployment scenarios. Our findings have interesting applications to the design and evaluation of multi-hop wireless networks.

Keywords: Multi-hop Wireless networks, Effective Radius model, t-hop neighborhood.

1 Introduction

A group of wireless nodes that wish to communicate may self-organize into a multi-hop wireless network, i.e., an ad hoc network, a sensor network, or a mesh network. Each node has a limited transmission range that covers a disk centered at the node. Node u can receive the signal from node v if it is within the transmission range of the sender v. Otherwise, two nodes communicate through multi-hop wireless links by employing intermediate nodes as relay points.

The expected number of one-hop neighbors[4] per node, also known as the node degree, is a fundamental property of a multi-hop wireless network. It is usually associated with the connectivity of the network. This is particularly true for random graphs where the graph is almost certainly connected if its average node degree is above some threshold [1]. The indications of the node degree have been extensively studied in [2–4]. However, knowing such information is not enough to explore other important characteristics of multi-hop wireless networks. The expected number of t-hop neighbors, where $t > 1$, plays an equally important role in many application scenarios.

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[4] Nodes that can be reached via one hop.

For example, when estimating the control message overheads of routing protocols such as AODV [5] and DSR [6], which mainly utilize flooding with a bounded TTL value to perform route discoveries, it is helpful to have the knowledge of the number of reachable t-hop neighbors, where $t = 2, 3, 4, \cdots$. In addition, other classical problems like fault tolerant node deployment [7], topology control [8–10], and multi-path routing [11], can be better explored with the information of t-hop neighbors.

Recently, security provisioning in multi-hop networks has become a central concern. It is indicated that the number of t-hop neighbors plays an important role in assisting security protocol design and analysis. For instance, in order to enhance data confidentiality, Lou, Liu, and Fang [12] propose to deliver secret messages via multiple paths. As a result, precise information about t-hop neighbors is required. In [13], *multi-hop path reinforcement* technique is employed to strengthen the security of an established link key. Moreover, for authentication schemes using en-route filtering methods [14, 15] to filter out injected packets, pairwise keys need to be setup among nodes that are some t-hop away. Clearly, all these mechanisms require the estimated number of t-hop neighbors.

Nevertheless, calculating the expected number of t-hop neighbors, denoted as d_t, is a non-trivial problem due to the randomness of the node positions (as we shall see in Section 2). Chan, Perrig, and Song [13] claims that $d_t = \pi * (t^2 - (t-1)^2) * \phi$, where ϕ is the node density on the average. Intuitively, this results only gives the upper bound of d_t.

As a matter of fact, lacking a sound method to calculate d_t often becomes a barrier in the design and actual deployment of multi-hop wireless networks. Indeed, it is such a concern that has motivated our work. In this paper, we propose a novel *Effective Radius* model to calculate d_t, the expected number of t-hop neighbors under a uniform random node distribution.

We explore the problem of determining the number of t-hop neighbors using a probabilistic approach. Assuming all nodes have a fixed transmission range R. Given the node density ϕ, we derive a series of equations to calculate d_t and the t-hop effective radius R_e^t that can cover $\sum_{i=1}^{t} d_t$ number of nodes. The Effective Radius (ER) model is an analytical tool that can recursively compute R_e^t and therefore derive d_t for $t = 2, 3, \cdots$. This Effective Radius (ER) model is evaluated through extensive simulation studies. Our results indicate that the ER model is efficient and accurate for moderate and high density multi-hop networks. To the best of our knowledge, this is a pioneer work in deriving the expected number of t-hop neighbors. Our major contributions are two-fold.

1. We propose a novel analytical model, termed as *Effective Radius*, for calculating the expected number of t-hop neighbors in a multi-hop wireless network.
2. Extensive simulation studies on different network deployment models are presented in detail, demonstrating that the ER model can accurately estimate d_t with simple calculations.

The remaining parts of this paper are organized as follows. The *Effective Radius* model is proposed in Section 2 and evaluated through simulation studies in Section 3. We conclude our paper in Section 4.

2 Effective Radius Model

In this section, we present our *Effective Radius* (ER) model to identify the expected number of t-hop neighbors[5] (d_t) a wireless node may have, where $t = 2, 3, \cdots$, in a uniformly randomly deployed large-scale multi-hop wireless network. The notations listed in Table 1 are utilized throughout the whole model derivation procedure.

N	The total number of nodes
A	The area of the deployment region
R	Transmission range
r	The Euclidian distance between two arbitrary nodes
t	Hop count
R_t^e	The effective radius of the tth-hop
d_t	The number of t-hop neighbors
P_t^a	The conditional probability of a node being a t-hop neighbor

Table 1. Notations.

2.1 Network Model and Assumptions

Consider a multi-hop wireless network with (i) a uniform random node distribution on an average spatial sense, (ii) no inter-node interference (INI), and (iii) omnidirectional transmission from each node. For simplicity, we assume that the wireless nodes are distributed in a vast terrain (e.g., square or disk) such that boundary effects can be ignored.

Each node is assumed to transmit with a fixed radio power. Two nodes can communicate directly with each other if and only if they are no more than a distance R apart.

2.2 Model Derivation

We assume that there are N nodes uniformly distributed in the deployment region with an area of A. Thus, an arbitrary node u can cover $d_1 = \pi R^2 \frac{N}{A} - 1$ nodes within one hop on the average. Now, how to derive d_t, where $t = 2, 3, \cdots$, the expected number of t-hop neighbors that an arbitrary node may have? In the next, we introduce our ER model to recursively compute these values.

In the first place, we consider the case of $t = 2$. Let v be another arbitrary node whose Euclidian distance to u is denoted by r. If $r \leq R$, v is a one-hop neighbor of u; if $r > 2R$, v can not be reached by u via two hops. Therefore, v is a two-hop neighbor of u if and only if (i) $R < r \leq 2R$ and (ii) u and v share at least one immediate neighbor[6].

Let E_1 be the event that the distance $r \in (R, 2R]$. Let E_2 be the event that u and v have at least one common immediate neighbor. Given $R < r \leq 2R$, $Pr[E_2|E_1]$ equals

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[5] in shortest path.

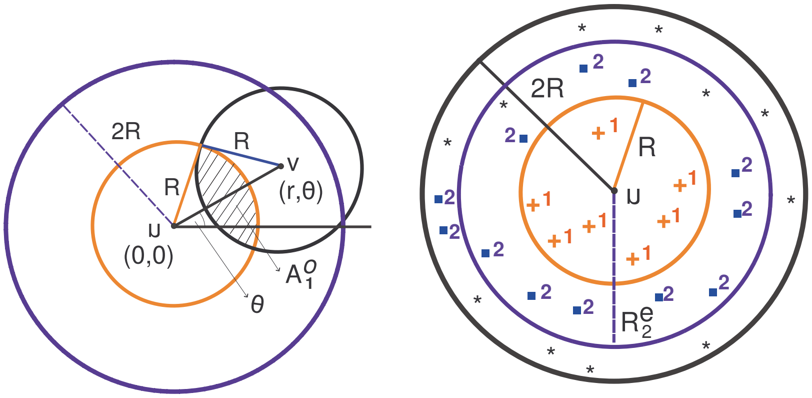
[6] A one-hop neighbor can also be called an immediate neighbor.

to the probability that at least one of the d_1 immediate neighbors of u falls into the region covered by both u and v. This overlapping area is denoted by A_1^o, as shown in Fig. 1(a). Thus, we have

$$Pr[E_2|E_1] = 1 - (1 - \frac{A_1^o}{\pi R^2})^{d_1}, \tag{1}$$

where A_1^o is defined as Eq. (2).

$$A_1^o = 2R^2 \arccos(\frac{r}{2R}) - \frac{r}{2}\sqrt{4R^2 - r^2}. \tag{2}$$



(a) v is a two-hop neighbor of u. (b) The effective two-hop radius R_2^e of u.

Fig. 1. The two-hop neighborhood and the effective two-hop neighborhood of a node u.

Based on Eq. (1), the expected value of $Pr[E_2|E_1]$ throughout the annulus region from R to $2R$ (see Fig. 1(a)), denoted by P_2^a, can be represented by

$$P_2^a = \frac{\int_0^{2\pi} d\theta \int_R^{2R} (1 - (1 - \frac{A_1^o}{\pi R^2})^{d_1}) r dr}{\pi((2R)^2 - R^2)}. \tag{3}$$

As a result, the number of u's two-hop neighbors, denoted by d_2, follows

$$d_2 = (3\pi R^2)\frac{N}{A}P_2^a. \tag{4}$$

Directly computing the exact number of t-hop neighbors is a difficult problem when t is larger than two. Therefore, we introduce the *Effective Radius* (ER) model to facilitate this computation. Let D_t be the expected number of neighbors that are at most t-hop away. In our ER model, the effective radius of the t-hop coverage of a node is defined as the radius of a virtual disk centered at the node that can cover D_t number of nodes.

For example, Fig. 1(b) depicts the effective radius for the case of two hops. In this figure, the virtual disk centered at u with a radius of R_2^e covers $d_1 + d_2$ number of nodes in total. These covered nodes include all the one-hop neighbors (labelled with plus signs), a number of two-hop neighbors (labelled with dots), and a few other nodes (labelled with star signs). Note that the number of two-hop neighbors that fall out of the virtual disk equals to the number of nodes that can't be reached from u within two hops but fall into this virtual disk.

Accordingly, the effective radius R_2^e for the two-hop case can be calculated as follows.

$$\pi(R_2^e)^2\frac{N}{A} = d_1 + d_2 + 1. \tag{5}$$

Plug $d_1 = \pi R^2 \frac{N}{A} - 1$ and Eq. (4) into Eq. (5), we obtain

$$R_2^e = \sqrt{R^2 + 3R^2 P_2^a}. \tag{6}$$

Now we are ready to derive the number of three-hop neighbors of u. In our ER model, v's transmission range remains to be R while u's transmission range is set to be R_2^e. In other words, the virtual disk with a radius R_2^e centered at u represents u's two-hop coverage. In this case, v is a three-hop neighbor of u if and only if (i) $R_2^e < r \le R_2^e + R$ and (ii) u's virtual disk covers at least one of v's immediate neighbors.

Let P_3^a be the probability that v is a three-hop neighbor of u given that the distance between u and v is in the range of $(R_2^e, R_2^e + R]$. With a similar analysis, we obtain

$$P_3^a = \frac{\int_0^{2\pi} d\theta \int_{R_2^e}^{(R_2^e+R)} (1 - (1 - \frac{A_2^o}{\pi R^2})^{d_1}) r dr}{\pi((R_2^e + R)^2 - (R_2^e)^2)}, \tag{7}$$

where A_2^o, the overlapping area covered by both u and v as shown in Fig 2(a), is regulated by Eq. (8).

$$\begin{aligned} A_2^o = {} & R^2 \arccos(\frac{r^2 + R^2 - (R_2^e)^2}{2rR}) \\ & + (R_2^e)^2 \arccos(\frac{r^2 + (R_2^e)^2 - R^2}{2rR_2^e}) \\ & - \frac{1}{2}\sqrt{4r^2(R_2^e)^2 - (r^2 - R^2 + (R_2^e)^2)^2}. \end{aligned} \tag{8}$$

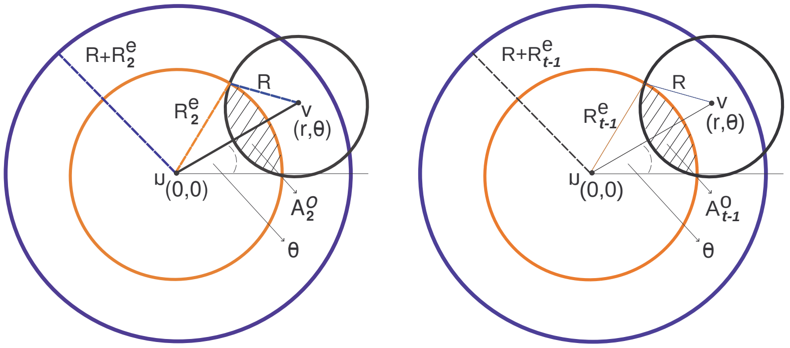
Thus, the number of u's three-hop neighbors can be approximated by

$$d_3 = \pi((R_2^e + R)^2 - (R_2^e)^2)\frac{N}{A}P_3^a. \tag{9}$$

And the equivalent radius for three hops is

$$R_3^e = \sqrt{(R_2^e)^2 + ((R_2^e + R)^2 - (R_2^e)^2)P_3^a}. \tag{10}$$

By recursively applying this procedure, we get the probability P_t^a of v being u's t-hop neighbor given that the Euclidean distance r between u and v satisfies $R_{t-1}^e < r \le$



(a) v is a three-hop neighbor of u. (b) v is a t-hop neighbor of u.

Fig. 2. Effective radius for t-hop neighbors.

$R_{t-1}^e + R$, the expected number of u's t-hop neighbors d_t, and the equivalent radius R_t^e as follows.

$$P_t^a = \frac{\int_0^{2\pi} d\theta \int_{R_{t-1}^e}^{(R_{t-1}^e+R)} (1-(1-\frac{A_{t-1}^o}{\pi R^2})^{d_1}) r dr}{\pi((R_{t-1}^e+R)^2-(R_{t-1}^e)^2)}, \tag{11}$$

$$d_t = \pi((R_{t-1}^e+R)^2-(R_{t-1}^e)^2)\frac{N}{A}P_t^a, \tag{12}$$

and

$$R_t^e = \sqrt{(R_{t-1}^e)^2 + ((R_{t-1}^e+R)^2-(R_{t-1}^e)^2)P_t^a}. \tag{13}$$

Our ER model will be validated through simulation studies in the following section.

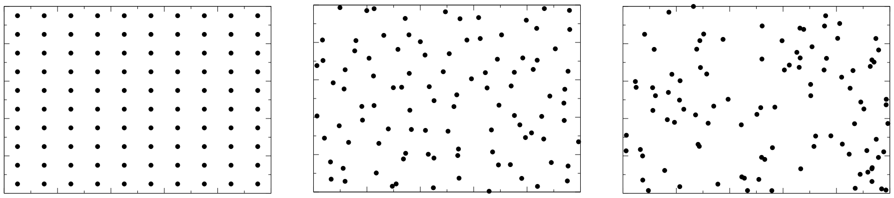
3 Evaluation of effective radius model

In this section, we evaluate the performance of our ER model through simulation studies that complement our analysis.

Network Model We consider a square terrain in which nodes are deployed according to the following three models (shown in Fig. 3):

- **Gird**: In a $\sqrt{N} \times \sqrt{N}$ grid deployment, each of the N nodes is located in the intersection of a grid. This is a deterministic placement.
- **Grid random**: Grid random deployment is similar to the grid deployment except that every node is placed in a uniform random manner within a grid instead of the intersection.
- **Uniform random**: In a uniform random deployment, each sensor falls at any location in the deployment area with an equal likelihood, independent of the other sensors.

The reason that we choose these three models is to examine the adaptivity of the ER model to different deployment scenarios given equal node density on the average.



(a) Grid. (b) Grid random. (c) Uniform random.

Fig. 3. Deployment models.

Validation Settings

- There are 6400 nodes to be deployed in a field of 80×80. Therefore, the node density $\phi = 1$, i.e., there is one node in a unit square on the average.
- We assume a disc-based transmission model where node has a transmission radius of R. By varying the transmission range R from 1.0 to 4.5 with a step of 0.5, we achieve the average node degree (the number of immediate neighbors) of $2.1, 6.1, 11.5, 18.6, 27.2, 37.5, 49.3$, and 62.6, respectively.
- The results are averaged over 1000 runs.

Validation Results The results are reported in Fig. 4. Based on this study, we draw the following conclusions.

- In almost all cases, the deterministic grid model performs the worst in accuracy compared to other models. This is particularly true when the node degree is low (e.g. Fig. 4(a)). The reason is because our ER model is derived under a probabilistic deployment assumption.
- The ER model does not give accurate results when the node degree is below certain threshold, as shown in Fig. 4(a). As a matter of fact, when the node degree is less than 6, the whole network tends to become disconnected in simulation, which is consistent with [2].
- The results of the ER model approach towards those of the simulation when the node degree becomes larger, as illustrated by Fig. 4(b) to Fig. 4(h). The higher the node degree, the more accurate the ER model. When the transmission range $R \geq 4.0$, as shown in Fig. 4(g) and Fig. 4(h), the largest difference of the results obtained from the ER model and those of the simulation is below 3.5%.
- The ER model is accurate and suitable for multi-hop networks that are uniformly and densely deployed.

4 Conclusions

This work was motivated by a fundamental problem emerged from the design and deployment of wireless multi-hop networks. In this paper, we derive an *Effective Radius* model to compute the estimated number of t-hop neighbors any node may have on the average. Simulation results demonstrate that this ER model can properly estimate t-hop neighbors in a densely and uniformly deployed network. Our findings have interesting applications in the design and evaluation of multi-hop wireless networks such as ad hoc networks, sensor networks, and mesh networks.

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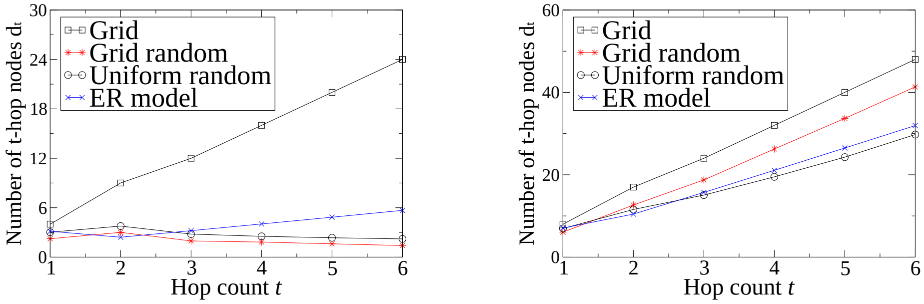
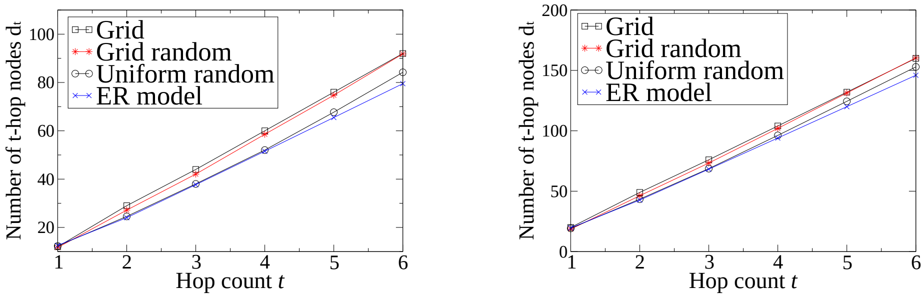
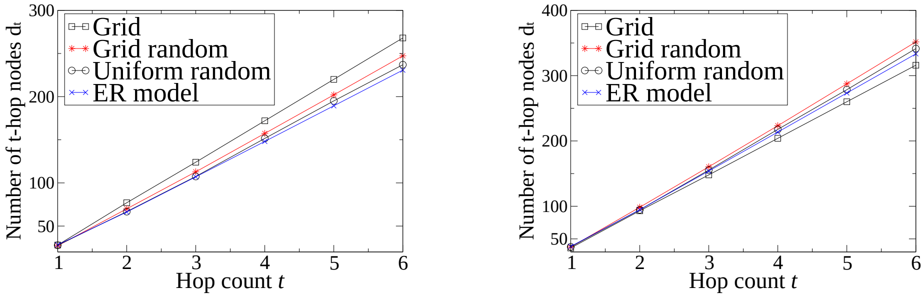


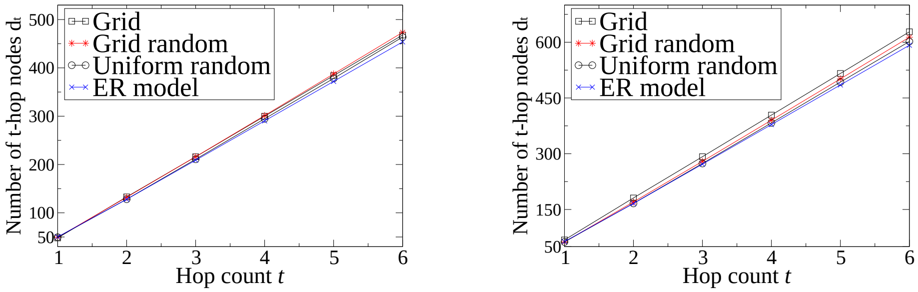
(a) $R = 1.0$, $d_1 = 2.1$.   (b) $R = 1.5$, $d_1 = 6.1$.



(c) $R = 2.0$, $d_1 = 11.5$.   (d) $R = 2.5$, $d_1 = 18.6$.



(e) $R = 3.0$, $d_1 = 27.2$.   (f) $R = 3.5$, $d_1 = 37.5$.



(g) $R = 4.0$, $d_1 = 49.3$.   (h) $R = 4.5$, $d_1 = 62.6$.

Fig. 4. Analytical results *vs.* simulation results.